

ISOMETRIC DILATIONS OF COMMUTING CONTRACTIONS AND BREHMER POSITIVITY

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ABSTRACT. It is well-known that an n -tuple ($n \geq 3$) of commuting contractions does not possess an isometric dilation, in general. Considering a class of n -tuple of commuting contractions satisfying certain positivity assumption, we construct their isometric dilations and consequently establish their von Neumann inequality. The positivity assumption is related to Brehmer positivity and motivated by the study of isometric dilations of operator tuples in [4].

1. INTRODUCTION

Starting point of the dilation theory is the result of Sz.-Nagy which says that a Hilbert space contraction always dilates to an isometry acting on a bigger Hilbert space. More precisely, Sz.-Nagy proved the following.

Theorem 1.1 ([19]). *Let T be a contraction on a Hilbert space \mathcal{H} . Then there exists a Hilbert space \mathcal{K} ($\mathcal{K} \supset \mathcal{H}$) and an isometry $V \in \mathcal{B}(\mathcal{K})$ such that*

$$T^k = P_{\mathcal{H}}V^k|_{\mathcal{H}}$$

for all $k \in \mathbb{Z}_+$, where $P_{\mathcal{H}}$ denotes the orthogonal projection in $\mathcal{B}(\mathcal{K})$ with range \mathcal{H} .

This result is the cornerstone of the extremely useful and extensive theory of Sz.-Nagy and Foias on single contractions [19]. Ando [3], generalizing Sz.-Nagy's dilation, constructed isometric dilations for pairs of commuting contractions. In other words, he constructed a pair of commuting isometries (V_1, V_2) on $\mathcal{K} \supseteq \mathcal{H}$ corresponding to each pair of commuting contractions (T_1, T_2) on \mathcal{H} such that

$$T_1^{m_1}T_2^{m_2} = P_{\mathcal{H}}V_1^{m_1}V_2^{m_2}|_{\mathcal{H}}$$

for all $(m_1, m_2) \in \mathbb{Z}_+^2$. Producing some counterexamples Parrott [21] showed that for $n \geq 3$, n -tuples of commuting contractions do not possess isometric dilations in general. This leads us to study n -tuples ($n \geq 3$) of commuting contractions more closely and identify n -tuples of commuting contractions which possess isometric dilations. For the rest of this article, we assume that $n \geq 3$. We denote by $\mathcal{T}^n(\mathcal{H})$ the set of all n -tuple of commuting contractions, that is

$$\mathcal{T}^n(\mathcal{H}) = \{(T_1, \dots, T_n) : T_i \in \mathcal{B}(\mathcal{H}), \|T_i\| \leq 1, T_i T_j = T_j T_i, 1 \leq i, j \leq n\},$$

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and for $T = (T_1, \dots, T_n) \in \mathcal{T}^n(\mathcal{H})$ and $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ we define $T^{\mathbf{k}} = T_1^{k_1} \dots T_n^{k_n}$. For Hilbert spaces \mathcal{H} and \mathcal{K} with $\mathcal{K} \supset \mathcal{H}$, an n -tuple of commuting isometries (respectively unitaries) $V \in \mathcal{T}^n(\mathcal{K})$ is called an isometric dilation (respectively unitary dilation) of $T \in \mathcal{T}^n(\mathcal{H})$ if

$$T^{\mathbf{k}} = P_{\mathcal{H}} V^{\mathbf{k}}|_{\mathcal{H}}$$

for all $\mathbf{k} \in \mathbb{Z}_+^n$. There is a growing literature exploring the interplay between dilation and positivity, and several classes of operator tuples under certain positivity assumptions are known to have isometric dilations. See [1], [7], [14], [20], [10], [11], [15], [12], [13], [4] and [5] for more details in the polydisc setup. The one which is relevant for us is the dilation of operator tuples under Brehmer positivity [7]. An n -tuple of commuting contraction $T \in \mathcal{T}^n(\mathcal{H})$ is said to satisfy Brehmer positivity if

$$\sum_{F \subset G} (-1)^{|F|} T_F T_F^* \geq 0$$

for all $G \subset \{1, \dots, n\}$, where $T_F := T_{n_1} \dots T_{n_k}$ for any $F = \{n_1, \dots, n_k\} \subset \{1, \dots, n\}$ and by convention $T_{\emptyset} = I_{\mathcal{H}}$. We denote by $\mathfrak{B}^n(\mathcal{H})$ the class of n -tuples of commuting contractions on \mathcal{H} satisfying Brehmer positivity, that is

$$\mathfrak{B}^n(\mathcal{H}) := \{T \in \mathcal{T}^n(\mathcal{H}) : T \text{ satisfies Brehmer positivity}\}.$$

It is clear from the definition that if $T \in \mathfrak{B}^n(\mathcal{H})$ then

$$(T_{n_1}, \dots, T_{n_k}) \in \mathfrak{B}^k(\mathcal{H})$$

for any non-empty subset $\{n_1, \dots, n_k\}$ of $\{1, \dots, n\}$. It has been shown in [7] that every $T \in \mathfrak{B}^n(\mathcal{H})$ possesses isometric dilation (also see [19]). In fact, more stronger result is true. Namely, $T \in \mathfrak{B}^n(\mathcal{H})$ if and only if T has a $*$ -regular unitary dilation U , that is

$$T^{\alpha_+} T^{*\alpha_-} = P_{\mathcal{H}} U^{*\alpha_-} U^{\alpha_+}|_{\mathcal{H}}$$

for all $\alpha \in \mathbb{Z}^n$, where $\alpha_+ = (\alpha_1^+, \dots, \alpha_n^+)$, $\alpha_- = (\alpha_1^-, \dots, \alpha_n^-) \in \mathbb{Z}_+^n$ and $\alpha_i^+ := \max\{\alpha_i, 0\}$, $\alpha_i^- := \max\{-\alpha_i, 0\}$. The aim of this article is to exhibit a class of n -tuple of commuting contractions having isometric dilations so that the class is larger than the class of n -tuples of commuting contractions satisfying Brehmer positivity. To describe the class of operator tuples succinctly, we adopt the following notation. For $T = (T_1, \dots, T_n) \in \mathcal{T}^n(\mathcal{H})$ and $1 \leq i \leq n$, we define

$$\hat{T}_i := (T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n) \in \mathcal{T}^{n-1}(\mathcal{H}),$$

the $(n-1)$ -tuple obtained from T by deleting T_i . The class we consider in this article is denoted by $\mathfrak{B}_{p,q}^n(\mathcal{H})$ for some $1 \leq p < q \leq n$ and defined as

$$\mathfrak{B}_{p,q}^n(\mathcal{H}) := \{T = (T_1, \dots, T_n) \in \mathcal{T}^n(\mathcal{H}) : \hat{T}_p, \hat{T}_q \in \mathfrak{B}^{n-1}(\mathcal{H})\}.$$

From the definition it is clear that $\mathfrak{B}^n(\mathcal{H}) \subset \mathfrak{B}_{p,q}^n(\mathcal{H})$ and the containment can be shown to be proper. By means of an explicit construction, we show that $T \in \mathfrak{B}_{p,q}^n(\mathcal{H})$ if and only if T has an isometric dilation V such that \hat{V}_p and \hat{V}_q are $*$ -regular isometric dilation of \hat{T}_p and \hat{T}_q , respectively. For the base case ($n=3$), the existence of isometric dilations for $\mathfrak{B}_{1,3}^3(\mathcal{H})$ is obtained by Gasper and Suciú in [15, Theorem 12]. However our proof, for this particular case, is completely different than that of [15] and is based on an explicit construction of

dilating isometries. The present consideration is also motivated by [4] where the authors considered the following class of operator tuples

$$(1.1) \quad \mathcal{T}_{p,q}^n := \{T \in \mathcal{T}^n(\mathcal{H}) : \hat{T}_p \text{ and } \hat{T}_q \text{ satisfy Szegő positivity and } \hat{T}_q \text{ is pure}\}$$

and found their isometric dilations explicitly. We say that $T \in \mathcal{T}^n(\mathcal{H})$ satisfies Szegő positivity if

$$\sum_{F \subset \{1, \dots, n\}} (-1)^{|F|} T_F T_F^* \geq 0,$$

and we say T is pure if $T_i^{*m} h \rightarrow 0$ as $m \rightarrow \infty$ for all $h \in \mathcal{H}$ and $i = 1, \dots, n$. It is easy to see that if $T \in \mathcal{T}^n(\mathcal{H})$ is pure and satisfies Szegő positivity then it satisfies Brehmer positivity. Thus it is evident that if $T \in \mathcal{T}_{p,q}^n(\mathcal{H})$ and T is pure then $T \in \mathfrak{B}_{p,q}^n(\mathcal{H})$. From this point of view, the present work is also a generalization of [4].

An added benefit of this consideration is the von Neumann inequality for the class $\mathfrak{B}_{p,q}^n(\mathcal{H})$. If $T \in \mathcal{T}^n(\mathcal{H})$ has an isometric dilation, then it is immediate that T satisfies the von Neumann inequality, that is for all $p \in \mathbb{C}[z_1, \dots, z_n]$

$$\|p(T)\| \leq \sup_{z \in \mathbb{D}^n} |p(z)|,$$

where \mathbb{D}^n is the open unit n -polydisc in \mathbb{C}^n . Thus, as an immediate consequence of our isometric dilations, we obtain that each tuple in $\mathfrak{B}_{p,q}^n(\mathcal{H})$ satisfies von Neumann inequality. It is worth mentioning here that von Neumann inequality does not hold in general for n -tuple of commuting contractions (see [24] and [9]). More details on von Neumann inequality for n -tuple of commuting contractions can be found in [8], [16], [17], [18] and [22].

Rest of the paper is organized as follows. In the next section we develop some background material and state some known results which are relevant in the present context. Section 3 deals with isometric dilations and von Neumann inequality for the class $\mathfrak{B}_{p,q}^n(\mathcal{H})$.

2. PRELIMINARIES

In most of the cases, isometric dilations of tuple of commuting contractions are isometric co-extensions. However, we use co-extension as an intermediate step to obtain isometric dilations in the present context. Let \mathcal{H} and \mathcal{K} be Hilbert spaces, and let $T \in \mathcal{T}^n(\mathcal{H})$ and V be an n -tuple of contractions on \mathcal{K} . We say that V is a *co-extension* of T if there exists an isometry $\Pi : \mathcal{H} \rightarrow \mathcal{K}$ such that

$$(2.1) \quad \Pi T_i^* = V_i^* \Pi,$$

for all $i = 1, \dots, n$. In addition, if

$$\mathcal{K} = \overline{\text{span}}\{V^{\mathbf{k}} h : \mathbf{k} \in \mathbb{Z}_+^n, h \in \text{ran } \Pi\},$$

then we say V is a *minimal co-extension* of T . We warn the reader here that we do not require V to be a commuting tuple. Let $\mathcal{Q} = \text{ran } \Pi$. Then, by equation (2.1),

$$\Pi T_i \Pi^* = (\Pi T_i \Pi^*)(\Pi \Pi^*) = (\Pi \Pi^* V_i)(\Pi \Pi^*) = P_{\mathcal{Q}} V_i|_{\mathcal{Q}}.$$

This implies that $(P_{\mathcal{Q}}V_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}V_n|_{\mathcal{Q}})$ is a commuting tuple of contractions even if V is not an commuting tuple of contractions. Moreover, (T_1, \dots, T_n) is unitary equivalent to $(P_{\mathcal{Q}}V_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}V_n|_{\mathcal{Q}})$. Also,

$$(P_{\mathcal{Q}}V_i|_{\mathcal{Q}})^* = \Pi T_i^* \Pi^* = V_i^* \Pi \Pi^* = V_i^*|_{\mathcal{Q}},$$

which implies, (V_1, \dots, V_n) is a co-extension of $(P_{\mathcal{Q}}V_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}V_n|_{\mathcal{Q}})$ ($\cong (T_1, \dots, T_n)$).

For a Hilbert space \mathcal{E} , the \mathcal{E} -valued Hardy space over \mathbb{D}^n is denoted by $H_{\mathcal{E}}^2(\mathbb{D}^n)$ and defined as the space of all \mathcal{E} -valued analytic functions $f = \sum_{\mathbf{k} \in \mathbb{Z}_+^n} a_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$ ($a_{\mathbf{k}} \in \mathcal{E}$) on \mathbb{D}^n such that

$$\sum_{\mathbf{k} \in \mathbb{Z}_+^n} \|a_{\mathbf{k}}\|^2 < \infty.$$

The space $H_{\mathcal{E}}^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space with kernel $\mathbb{S}_n I_{\mathcal{E}}$ where \mathbb{S}_n is the Szegő kernel on the polydisc \mathbb{D}^n given by

$$\mathbb{S}_n(\mathbf{z}, \mathbf{w}) = \prod_{i=1}^n (1 - z_i \bar{w}_i)^{-1} \quad (\mathbf{z}, \mathbf{w} \in \mathbb{D}^n).$$

The n -tuple of shifts $(M_{z_1}, \dots, M_{z_n})$ on $H_{\mathcal{E}}^2(\mathbb{D}^n)$ is defined by

$$(M_{z_i} f)(\mathbf{w}) = w_i f(\mathbf{w}) \quad (\mathbf{w} \in \mathbb{D}^n),$$

for all $i = 1, \dots, n$ and is a tuple of commuting isometries. For $T \in \mathcal{T}^n(\mathcal{H})$, we say $T \in \mathcal{T}^n(\mathcal{H})$ satisfies Szegő positivity if $\mathbb{S}_n^{-1}(T, T^*) \geq 0$, where

$$\mathbb{S}_n^{-1}(T, T^*) := \sum_{F \subset \{1, \dots, n\}} (-1)^{|F|} T_F T_F^*.$$

In such a case, we define the defect operator and the defect spaces as

$$D_T := \mathbb{S}_n^{-1}(T, T^*)^{1/2} \quad \text{and} \quad \mathcal{D}_T := \overline{\text{ran}} \mathbb{S}_n^{-1}(T, T^*),$$

respectively. The map $\Pi : \mathcal{H} \rightarrow H_{\mathcal{D}_T}^2(\mathbb{D}^n)$ defined by

$$(2.2) \quad (\Pi h)(\mathbf{z}) = \sum_{\mathbf{k} \in \mathbb{Z}_+^n} \mathbf{z}^{\mathbf{k}} \otimes D_T T^{*\mathbf{k}} h \quad (\mathbf{z} \in \mathbb{D}^n)$$

is called the *canonical dilation map* corresponding to a Szegő tuple $T \in \mathcal{T}^n(\mathcal{H})$ and it satisfies

$$\Pi T_i^* = M_{z_i}^* \Pi$$

for all $i = 1, \dots, n$, and

$$\|\Pi h\|^2 = \lim_{\mathbf{k} \rightarrow \infty} \sum_{F \subset \{1, \dots, n\}} (-1)^{|F|} \|T_F^{*\mathbf{k}} h\|^2$$

for all $h \in \mathcal{H}$. In addition, if T is pure then by the above identity Π is an isometry and therefore the n -tuple of shift $(M_{z_1}, \dots, M_{z_n})$ on $H_{\mathcal{D}_T}^2(\mathbb{D}^n)$ is a co-extension of T . In particular, a pure Szegő n -tuple T dilates to the n -tuple of shifts on $H_{\mathcal{D}_T}^2(\mathbb{D}^n)$ (see [10] and [20]). A pure Szegő tuple $T \in \mathcal{T}^n(\mathcal{H})$ is also a member of $\mathfrak{B}^n(\mathcal{H})$, that is T satisfies Brehmer positivity. Thus

the class $\mathfrak{B}^n(\mathcal{H})$ is larger than the class of pure and Szegő n -tuples on \mathcal{H} . Recall that an isometric dilation $V \in \mathcal{T}^n(\mathcal{K})$ of $T \in \mathcal{T}^n(\mathcal{H})$ is $*$ -regular (respectively regular) if it satisfies

$$P_{\mathcal{H}}V^{*\alpha-}V^{\alpha+}|_{\mathcal{H}} = T^{\alpha+}T^{*\alpha-} \quad (\alpha \in \mathbb{Z}^n)$$

or respectively,

$$P_{\mathcal{H}}V^{*\alpha-}V^{\alpha+}|_{\mathcal{H}} = T^{*\alpha-}T^{\alpha+} \quad (\alpha \in \mathbb{Z}^n).$$

It is well-known that $T \in \mathfrak{B}^n(\mathcal{H})$ if and only if T has a $*$ -regular isometric dilation (see [19]). For explicit constructions of $*$ -regular isometric dilations for the class $\mathfrak{B}^n(\mathcal{H})$ see [2] and [23]. Simply taking adjoint, it is evident that $T^* \in \mathfrak{B}^n(\mathcal{H})$ if and only if T has a regular isometric dilation. For example, if (V, T) is a pair of commuting contractions on \mathcal{H} and if V is an isometry then $(V^*, T^*) \in \mathfrak{B}^2(\mathcal{H})$ and therefore (V, T) has a regular isometric dilation. We end the section with a lemma which will be useful for us. The lemma may be known to experts and we are unable to find any suitable reference for it, and that is why we include a proof. We say that an operator tuple $T \in \mathcal{T}^n(\mathcal{H})$ is doubly commuting if $T_i T_j^* = T_j^* T_i$ for all $1 \leq i < j \leq n$.

Lemma 2.1. *Let (W_1, W_2) on \mathcal{K} be a minimal regular unitary dilation of (T_1, T_2) on \mathcal{H} . Suppose (S_1, \dots, S_d) is a tuple of isometries on \mathcal{H} such that (T_1, S_1, \dots, S_d) and (T_2, S_1, \dots, S_d) are doubly commuting. Then there exists a commuting tuple of isometries (U_1, \dots, U_d) on \mathcal{K} such that $(W_1, W_2, U_1, \dots, U_d)$ is an isometric dilation of the tuple $(T_1, T_2, S_1, \dots, S_d)$ and that (W_1, U_1, \dots, U_d) and (W_2, U_1, \dots, U_d) are doubly commuting.*

Moreover, (U_1, \dots, U_d) is both an extension and a co-extension of (S_1, \dots, S_d) .

Proof. Let $W = (W_1, W_2)$. Since W on \mathcal{K} is the minimal unitary dilation of $T = (T_1, T_2)$,

$$(2.3) \quad \mathcal{K} = \overline{\text{span}}\{W^{\mathbf{k}}h : \mathbf{k} \in \mathbb{Z}^2, h \in \mathcal{H}\}.$$

For $1 \leq i \leq d$, $h_1, \dots, h_r \in \mathcal{H}$ and $\mathbf{k}^1, \dots, \mathbf{k}^r \in \mathbb{Z}^2$, we have

$$\begin{aligned} \left\| \sum_{l=1}^r W^{\mathbf{k}^l} S_i h_l \right\|^2 &= \sum_{l,m=1}^r \langle W^{*\mathbf{k}^m} W^{\mathbf{k}^l} S_i h_l, S_i h_m \rangle \\ &= \sum_{l,m=1}^r \langle W^{*(\mathbf{k}^l - \mathbf{k}^m)} W^{(\mathbf{k}^l - \mathbf{k}^m)} S_i h_l, S_i h_m \rangle \\ &= \sum_{l,m=1}^r \langle T^{*(\mathbf{k}^l - \mathbf{k}^m)} T^{(\mathbf{k}^l - \mathbf{k}^m)} S_i h_l, S_i h_m \rangle \quad [\text{by regular dilation}] \\ &= \sum_{l,m=1}^r \langle T^{*(\mathbf{k}^l - \mathbf{k}^m)} T^{(\mathbf{k}^l - \mathbf{k}^m)} h_l, h_m \rangle, \quad [\text{by doubly commuting property}] \end{aligned}$$

and by a similar calculation we also have

$$\left\| \sum_{l=1}^r W^{\mathbf{k}^l} h_l \right\|^2 = \sum_{l,m=1}^r \langle T^{*(\mathbf{k}^l - \mathbf{k}^m)} T^{(\mathbf{k}^l - \mathbf{k}^m)} h_l, h_m \rangle.$$

This shows that for all $h_1, \dots, h_r \in \mathcal{H}$ and $\mathbf{k}^1, \dots, \mathbf{k}^r \in \mathbb{Z}^2$,

$$\left\| \sum_{l=1}^r W^{\mathbf{k}^l} S_i h_l \right\|^2 = \left\| \sum_{l=1}^r W^{\mathbf{k}^l} h_l \right\|^2$$

for any $1 \leq i \leq d$. Hence, by the minimality of the unitary dilation (2.3), we have isometry $U_i : \mathcal{K} \rightarrow \mathcal{K}$ defined by

$$U_i(W^{\mathbf{k}}h) = W^{\mathbf{k}}S_i h \quad (h \in \mathcal{H}, \mathbf{k} \in \mathbb{Z}^2),$$

for all $i = 1, \dots, d$. It is easy to see that

$$U_i|_{\mathcal{H}} = S_i, \quad U_i U_j = U_j U_i \text{ and } U_i W_m = W_m U_i$$

for all $i, j = 1, \dots, d$ and $m = 1, 2$. Consequently, for all $\alpha \in \mathbb{Z}_+^2$ and for all $(m_1, \dots, m_d) \in \mathbb{Z}_+^d$,

$$P_{\mathcal{H}} W^\alpha U_1^{m_1} \dots U_d^{m_d} |_{\mathcal{H}} = P_{\mathcal{H}} W^\alpha |_{\mathcal{H}} (S_1^{m_1} \dots S_d^{m_d}) = T^\alpha S_1^{m_1} \dots S_d^{m_d}.$$

Thus $(W_1, W_2, U_1, \dots, U_d)$ is an isometric dilation of $(T_1, T_2, S_1, \dots, S_d)$. It remains to show that (U_1, \dots, U_d) is a co-extension of (S_1, \dots, S_d) . To this end, for all $h, h' \in \mathcal{H}$ and $\mathbf{k} \in \mathbb{Z}^2$,

$$\langle U_i^* h, W^{\mathbf{k}} h' \rangle = \langle h, P_{\mathcal{H}} W^{\mathbf{k}} S_i h' \rangle = \langle h, P_{\mathcal{H}} W^{*\mathbf{k}-} W^{\mathbf{k}+} S_i h' \rangle = \langle h, T^{*\mathbf{k}-} T^{\mathbf{k}+} S_i h' \rangle, \quad (1 \leq i \leq d)$$

and using the fact that S_i doubly commutes with T , we have

$$\langle h, T^{*\mathbf{k}-} T^{\mathbf{k}+} S_i h' \rangle = \langle h, S_i T^{*\mathbf{k}-} T^{\mathbf{k}+} h' \rangle = \langle S_i^* h, P_{\mathcal{H}} W^{*\mathbf{k}-} W^{\mathbf{k}+} h' \rangle = \langle S_i^* h, W^{\mathbf{k}} h' \rangle \quad (1 \leq i \leq d).$$

Thus, $U_i^* h = S_i^* h$, for all $h \in \mathcal{H}$ and for all $i = 1, \dots, d$. Hence the proof follows. \square

3. ISOMETRIC DILATIONS AND VON NEUMANN INEQUALITY FOR $\mathfrak{B}_{p,q}^n(\mathcal{H})$

The aim of this section is to construct isometric dilations for the class $\mathfrak{B}_{p,q}^n(\mathcal{H})$ for some $1 \leq p < q \leq n$. For simplicity we fix $p = 1, q = n$ and construct isometric dilations for the class $\mathfrak{B}_{1,n}^n(\mathcal{H})$. This simplification is harmless as by a suitable rearrangement any operator tuple in $\mathfrak{B}_{p,q}^n(\mathcal{H})$ can be viewed as a member of $\mathfrak{B}_{1,n}^n(\mathcal{H})$. First we introduce some notations which will be followed throughout the paper.

For $T \in \mathcal{T}^n(\mathcal{H})$ and for each non-empty order subset $G = \{j_1, \dots, j_r\}$ of $\{1, \dots, n\}$, we define

$$T(G) := (T_{j_1}, \dots, T_{j_r}).$$

With the above notation, if $T \in \mathfrak{B}^n(\mathcal{H})$ then $T(G) \in \mathfrak{B}^{|G|}(\mathcal{H})$ for any non-empty subset G of $\{1, \dots, n\}$, and therefore we define the corresponding defect operator and defect spaces as

$$D_{T,G} := \mathbb{S}_r^{-1}(T(G), T(G)^*)^{1/2} = \left(\sum_{F \subset G} (-1)^{|F|} T_F T_F^* \right)^{1/2} \quad \text{and} \quad \mathcal{D}_{T,G} := \overline{\text{ran}} D_{T,G}.$$

For $T \in \mathcal{T}^n(\mathcal{H})$, we also set

$$\hat{T}_{1n} := (T_1 T_n, T_2, \dots, T_{n-1}),$$

the $(n-1)$ -tuple in $\mathcal{T}^{n-1}(\mathcal{H})$ obtained from T by removing T_n and replacing T_1 by $T_1 T_n$. Next we observe a crucial property of \hat{T}_{1n} when $T \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. A similar result is also observed in [4, Lemma 5.1] in the context of Szegő positivity.

Lemma 3.1. *Let $T = (T_1, \dots, T_n) \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. Then $\hat{T}_{1n} = (T_1 T_n, T_2, \dots, T_{n-1}) \in \mathfrak{B}^{n-1}(\mathcal{H})$.*

Proof. First note that for $G \subset \{1, \dots, n-1\}$ with $1 \notin G$,

$$\sum_{F \subset G} (-1)^{|F|} (\hat{T}_{1n})_F (\hat{T}_{1n})_F^* = \sum_{F \subset G} (-1)^{|F|} (\hat{T}_n)_F (\hat{T}_n)_F^* \geq 0.$$

On the other hand, for $G \subset \{1, \dots, n-1\}$ with $1 \in G$, it can be checked that

$$(3.1) \quad \sum_{F \subset G} (-1)^{|F|} (\hat{T}_{1n})_F (\hat{T}_{1n})_F^* = \sum_{F \subset G} (-1)^{|F|} (\hat{T}_1)_F (\hat{T}_1)_F^* + T_n \left(\sum_{F \subset G} (-1)^{|F|} (\hat{T}_n)_F (\hat{T}_n)_F^* \right) T_n^* \geq 0.$$

This completes the proof. \square

The key observation in the above lemma is the identity (3.1) which we use repeatedly in this article. We rewrite the identity in terms of defect operators as follows. For $T \in \mathfrak{B}_{1,n}^n(\mathcal{H})$ and $G \subset \{1, \dots, n-1\}$ with $1 \in G$ we have

$$(3.2) \quad D_{\hat{T}_{1n}, G}^2 + T_1 D_{\hat{T}_1, G}^2 T_1^* = D_{\hat{T}_{1n}, G}^2 = D_{\hat{T}_1, G}^2 + T_n D_{\hat{T}_n, G}^2 T_n^*.$$

By the above lemma, if $T \in \mathfrak{B}_{1,n}^n(\mathcal{H})$ then $\hat{T}_{1n} \in \mathfrak{B}^{n-1}(\mathcal{H})$ and therefore for all $G = \{j_1, \dots, j_{|G|}\} \subset \{1, \dots, n-1\}$, $\hat{T}_{1n}(G) \in \mathfrak{B}^{|G|}(\mathcal{H})$. We denote the corresponding canonical dilation map, as in (2.2), of $\hat{T}_{1n}(G)$ by Π_G , that is $\Pi_G : \mathcal{H} \rightarrow H_{\mathcal{D}_{\hat{T}_{1n}, G}}^2(\mathbb{D}^{|G|})$ defined by

$$(3.3) \quad (\Pi_G h)(z) = \sum_{k \in \mathbb{Z}_+^{|G|}} z^k \otimes D_{\hat{T}_{1n}, G} \hat{T}_{1n}(G)^{*k} h \quad (h \in \mathcal{H}, z \in \mathbb{D}^{|G|})$$

such that

$$\Pi_G T_{j_i}^* = M_{z_i}^* \Pi_G$$

for all $i = 1, \dots, |G|$. In particular if $G = \{1 = j_1, \dots, j_{|G|}\}$, then

$$\Pi_G T_1^* T_n^* = M_{z_1}^* \Pi_G \quad \text{and} \quad \Pi_G T_{j_i}^* = M_{z_i}^* \Pi_G$$

for all $i = 2, \dots, |G|$. The next lemma will be the key to factorize M_{z_1} further as a product of M_Φ and M_Ψ so that we can split the first intertwining relation above into

$$\Pi_G T_1^* = M_\Phi^* \Pi_G \quad \text{and} \quad \Pi_G T_n^* = M_\Psi^* \Pi_G$$

where Φ and Ψ are inner multipliers on $\mathbb{D}^{|G|}$ with their product being z_1 . Such a pair of multipliers (Φ, Ψ) is known as BCL pair and it turns out that the only way one can factorize a shift is through BCL pair (see [6] and see [13] for its counterpart in the context of pure contractions).

Lemma 3.2. *Let $T = (T_1 = RS, T_2, \dots, T_n) \in \mathfrak{B}^n(\mathcal{H})$ for some commuting contractions $R, S \in \mathcal{B}(\mathcal{H})$ and let $G \subset \{1, \dots, n\}$. Suppose that there exist bounded operators F_1 and F_2 on \mathcal{H} , Hilbert spaces \mathcal{F}_1 and \mathcal{F}_2 with $\mathcal{F}_i \supseteq \overline{\text{ran}} F_i$ ($i = 1, 2$), a Hilbert space \mathcal{E} , an isometry $\Gamma_G : \mathcal{D}_{T, G} \rightarrow \mathcal{E}$ and unitaries*

$$U_i = \begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} : \mathcal{E} \oplus \mathcal{F}_i \rightarrow \mathcal{E} \oplus \mathcal{F}_i \quad (i = 1, 2)$$

satisfying

$$U_1(\Gamma_G D_{T,G} h, F_1 R^* S^* h) = (\Gamma_G D_{T,G} R^* h, F_1 h)$$

and

$$U_2(\Gamma_G D_{T,G} h, F_2 R^* S^* h) = (\Gamma_G D_{T,G} S^* h, F_2 h)$$

for all $h \in \mathcal{H}$, then

$$(I \otimes \Gamma_G) \Pi_G R^* = M_{\Phi_1}^*(I \otimes \Gamma_G) \Pi_G \quad \text{and} \quad (I \otimes \Gamma_G) \Pi_G S^* = M_{\Phi_2}^*(I \otimes \Gamma_G) \Pi_G$$

where $\Phi_i(\mathbf{z}) = A_i^* + z_1 C_i^* B_i^*$ ($\mathbf{z} \in \mathbb{D}^{|\mathcal{G}|}$) is the transfer function of the unitary U_i^* for all $i = 1, 2$ and Π_G is the canonical dilation map of $T(G)$, as in (3.3).

Proof. Because of the symmetric roles of R and S , we only prove that $(I \otimes \Gamma_G) \Pi_G R^* = M_{\Phi_1}^*(I \otimes \Gamma_G) \Pi_G$. Let $\mathbf{l} \in \mathbb{Z}_+^{|\mathcal{G}|}$ and $\eta \in \mathcal{E}$. Then for all $h \in \mathcal{H}$

$$\begin{aligned} \langle (I \otimes \Gamma_G) \Pi_G R^* h, \mathbf{z}^{\mathbf{l}} \otimes \eta \rangle &= \langle (I \otimes \Gamma_G) \sum_{\mathbf{k} \in \mathbb{Z}_+^{|\mathcal{G}|}} \mathbf{z}^{\mathbf{k}} \otimes D_{T,G} T(G)^{*\mathbf{k}} R^* h, \mathbf{z}^{\mathbf{l}} \otimes \eta \rangle \\ &= \langle \Gamma_G D_{T,G} T(G)^{*\mathbf{l}} R^* h, \eta \rangle. \end{aligned}$$

Since

$$U_1(\Gamma_G D_{T,G} h, F_1 R^* S^* h) = (\Gamma_G D_{T,G} R^* h, F_1 h), \quad (h \in \mathcal{H})$$

we have

$$\Gamma_G D_{T,G} R^* = A_1 \Gamma_G D_{T,G} + B_1 F_1 R^* S^*, \quad \text{and} \quad F_1 = C_1 \Gamma_G D_{T,G}.$$

Combining these together, we get

$$\Gamma_G D_{T,G} R^* = A_1 \Gamma_G D_{T,G} + B_1 C_1 \Gamma_G D_{T,G} R^* S^*.$$

This implies that

$$\begin{aligned} \langle M_{\Phi_1}^*(I \otimes \Gamma_G) \Pi_G h, \mathbf{z}^{\mathbf{l}} \otimes \eta \rangle &= \langle (I \otimes \Gamma_G) \Pi_G h, M_{\Phi_1}(\mathbf{z}^{\mathbf{l}} \otimes \eta) \rangle \\ &= \langle (I \otimes \Gamma_G) \sum_{\mathbf{k} \in \mathbb{Z}_+^{|\mathcal{G}|}} \mathbf{z}^{\mathbf{k}} \otimes D_{T,G} T(G)^{*\mathbf{k}} h, (A_1^* + z_1 C_1^* B_1^*)(\mathbf{z}^{\mathbf{l}} \otimes \eta) \rangle \\ &= \langle (A_1 \Gamma_G D_{T,G} + B_1 C_1 \Gamma_G D_{T,G} R^* S^*) T(G)^{*\mathbf{l}} h, \eta \rangle \\ &= \langle \Gamma_G D_{T,G} T(G)^{*\mathbf{l}} R^* h, \eta \rangle. \end{aligned}$$

Thus $(I \otimes \Gamma_G) \Pi_G R^* = M_{\Phi_1}^*(I \otimes \Gamma_G) \Pi_G$. This completes the proof. \square

Our construction of isometric dilations for tuples in $\mathfrak{B}_{1,n}^n(\mathcal{H})$ relies on getting co-extensions of the tuples first. On the other hand, construction of these co-extensions is build upon obtaining certain operator tuples corresponding to each $G \subset \{1, \dots, n-1\}$. In the next two lemmas we construct these operator tuples. The first lemma deals with the case when $1 \in G$.

Lemma 3.3. *Let $T = (T_1, \dots, T_n) \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. Let $G = \{m_1, \dots, m_r\} \subset \{1, \dots, n-1\}$ with $1 = m_1 \in G$ and set $\bar{G} = \{1, \dots, n-1\} \setminus G$. Suppose that T_j is a co-isometry for all $j \in \bar{G}$ and Π_G be as in (3.3). Then there exist a Hilbert space \mathcal{H}_G , an isometry $\Gamma_G : \mathcal{D}_{\hat{T}_{1n,G}} \rightarrow \mathcal{H}_G$ and n -tuple of contractions $V = (V_1, \dots, V_n)$ on $H_{\mathcal{H}_G}^2(\mathbb{D}^r)$ such that \hat{V}_1 and \hat{V}_n are doubly commuting and*

$$(I \otimes \Gamma_G)\Pi_G T_i^* = V_i^*(I \otimes \Gamma_G)\Pi_G, \quad (1 \leq i \leq n)$$

where $\Pi_G : \mathcal{H} \rightarrow H_{\mathcal{D}_{\hat{T}_{1n,G}}}^2(\mathbb{D}^r)$ is the dilation map of $\hat{T}_{1n}(G)$ and

$$V_1 = P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_\Phi|_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)}, V_n = P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_\Psi|_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)}, V_{m_i} = M_{z_i} \quad (1 < i \leq r)$$

and

$$V_i = (I \otimes W_i), \quad (i \in \bar{G})$$

for inner functions Φ and Ψ , depend only on z_1 variable, in $H_{\mathcal{B}(\mathcal{K}_G)}^\infty(\mathbb{D}^r)$ with $\Phi(z)\Psi(z) = z_1$ and a Hilbert space \mathcal{K}_G containing \mathcal{H}_G and for unitaries W_i on \mathcal{H}_G .

Moreover, one also has

$$(I \otimes \Gamma_G)\Pi_G T_1^* = M_\Phi^*(I \otimes \Gamma_G)\Pi_G, \quad \text{and} \quad (I \otimes \Gamma_G)\Pi_G T_n^* = M_\Psi^*(I \otimes \Gamma_G)\Pi_G.$$

Proof. We first define several unitaries which will make a way to define the n -tuple of contractions (V_1, \dots, V_n) . Since T_j is a co-isometry, then

$$T_j D_{\hat{T}_{1,G}}^2 T_j^* = \sum_{F \subset G} (-1)^{|F|} (\hat{T}_1(G))_F T_j T_j^* (\hat{T}_1(G))_F^* = D_{\hat{T}_{1,G}}^2$$

and

$$T_j D_{\hat{T}_n,G}^2 T_j^* = \sum_{F \subset G} (-1)^{|F|} (\hat{T}_n(G))_F T_j T_j^* (\hat{T}_n(G))_F^* = D_{\hat{T}_n,G}^2$$

for all $j \in \bar{G}$. Then by Douglas' lemma, for all $j \in \bar{G}$, there exist co-isometries $W_{j,1} : \mathcal{D}_{\hat{T}_{1,G}} \rightarrow \mathcal{D}_{\hat{T}_{1,G}}$ and $W_{j,n} : \mathcal{D}_{\hat{T}_n,G} \rightarrow \mathcal{D}_{\hat{T}_n,G}$ such that

$$W_{j,1}^* D_{\hat{T}_{1,G}} h = D_{\hat{T}_{1,G}} T_j^* h \quad \text{and} \quad W_{j,n}^* D_{\hat{T}_n,G} h = D_{\hat{T}_n,G} T_j^* h \quad (h \in \mathcal{H}).$$

Let the commuting tuple of unitaries $\{W'_{j,n} : j \in \bar{G}\}$ on $\mathcal{K}_{n,G} \supseteq \mathcal{D}_{\hat{T}_n,G}$ be the minimal unitary co-extension of the tuple of commuting co-isometries $\{W_{j,n} : j \in \bar{G}\}$ and let the commuting tuple of unitaries $\{W'_{j,1} : j \in \bar{G}\}$ on $\mathcal{K}_{1,G} \supseteq \mathcal{D}_{\hat{T}_{1,G}}$ be the minimal unitary co-extension of the commuting tuple of co-isometries $\{W_{j,1} : j \in \bar{G}\}$. Set

$$W'_j := W'_{j,n} \oplus W'_{j,1} \quad (j \in \bar{G}) \quad \text{and} \quad \mathcal{K}'_G := \mathcal{K}_{n,G} \oplus \mathcal{K}_{1,G}.$$

Then the commuting tuple of unitaries $\{W'_j : j \in \bar{G}\}$ on \mathcal{K}'_G is the minimal unitary co-extension of $\{W_{j,n} \oplus W_{j,1} : j \in \bar{G}\}$. Now the identity, as noted in (3.2),

$$D_{\hat{T}_n,G}^2 + T_1 D_{\hat{T}_{1,G}}^2 T_1^* = D_{\hat{T}_{1n,G}}^2 = D_{\hat{T}_{1,G}}^2 + T_n D_{\hat{T}_n,G}^2 T_n^*$$

implies that there exist an isometry $\Gamma_G : \mathcal{D}_{\hat{T}_{1n,G}} \rightarrow \mathcal{K}'_G$ and a unitary

$$U' : \mathcal{Q}_G := \{(D_{\hat{T}_n,G} h, D_{\hat{T}_{1,G}} T_1^* h) : h \in \mathcal{H}\} \rightarrow \tilde{\mathcal{Q}}_G := \{(D_{\hat{T}_n,G} T_n^* h, D_{\hat{T}_{1,G}} h) : h \in \mathcal{H}\}$$

such that

$$(3.4) \quad \Gamma_G(D_{\hat{T}_{1n,G}}h) = (D_{\hat{T}_{1n,G}}h, D_{\hat{T}_{1,G}}T_1^*h), \quad (h \in \mathcal{H})$$

and

$$U'(D_{\hat{T}_{1n,G}}h, D_{\hat{T}_{1,G}}T_1^*h) = (D_{\hat{T}_{1n,G}}T_n^*h, D_{\hat{T}_{1,G}}h) \quad (h \in \mathcal{H}).$$

By the construction, note that \mathcal{Q}_G and $\tilde{\mathcal{Q}}_G$ are joint $\{W_j^* : j \in \bar{G}\}$ -invariant subspaces and a straightforward calculation shows that U' intertwines the tuple of isometries $\{W_j^*|_{\mathcal{Q}_G} : j \in \bar{G}\}$ and $\{W_j^*|_{\tilde{\mathcal{Q}}_G} : j \in \bar{G}\}$, that is

$$(3.5) \quad U'W_j^*|_{\mathcal{Q}_G} = (W_j^*|_{\tilde{\mathcal{Q}}_G})U' \quad (j \in \bar{G}).$$

Let $\mathcal{H}_G \subseteq \mathcal{K}'_G$ and $\tilde{\mathcal{H}}_G \subseteq \mathcal{K}'_G$ be the smallest joint $\{W_j^* : j \in \bar{G}\}$ reducing subspaces containing \mathcal{Q}_G and $\tilde{\mathcal{Q}}_G$, respectively. More precisely,

$$\mathcal{H}_G = \bigvee_{\mathbf{k} \in \mathbb{Z}_+^{|\bar{G}|}, q \in \mathcal{Q}_G} \prod_{j \in \bar{G}} W_j^{k_j} q \quad \text{and} \quad \tilde{\mathcal{H}}_G = \bigvee_{\mathbf{k} \in \mathbb{Z}_+^{|\bar{G}|}, \tilde{q} \in \tilde{\mathcal{Q}}_G} \prod_{j \in \bar{G}} W_j^{k_j} \tilde{q}.$$

Then $\{W_j^*|_{\mathcal{H}_G} : j \in \bar{G}\}$ and $\{W_j^*|_{\tilde{\mathcal{H}}_G} : j \in \bar{G}\}$ are the minimal unitary extension of $\{W_j^*|_{\mathcal{Q}_G} : j \in \bar{G}\}$ and $\{W_j^*|_{\tilde{\mathcal{Q}}_G} : j \in \bar{G}\}$, respectively. By a well-known intertwining lifting theorem, we extend U' to a unitary

$$U'' : \mathcal{H}_G \rightarrow \tilde{\mathcal{H}}_G \quad \text{satisfying} \quad U''|_{\mathcal{Q}_G} = U'$$

and

$$(3.6) \quad U''W_j^*|_{\mathcal{H}_G} = (W_j^*|_{\tilde{\mathcal{H}}_G})U'' = W_j^*U'' \quad (j \in \bar{G}).$$

By adding an infinite dimensional Hilbert space \mathcal{K} if necessary, we extend U'' further to get a unitary

$$(3.7) \quad U : \mathcal{K}_G \rightarrow \mathcal{K}_G \quad \text{such that} \quad U|_{\mathcal{H}_G} = U'',$$

where $\mathcal{K}_G := \mathcal{K}'_G \oplus \mathcal{K}$. We set

$$W_j := W_j^*|_{\mathcal{H}_G}$$

for all $j \in \bar{G}$. The stage is set and we now proceed to find the contractions with appropriate properties.

First, we define

$$V_j := I_{H^2(\mathbb{D}^r)} \otimes W_j \quad (j \in \bar{G}).$$

Then V_j is a unitary on $H_{\mathcal{H}_G}^2(\mathbb{D}^r)$ and observe that for all $h \in \mathcal{H}$,

$$\begin{aligned}
(I \otimes W_j^*)(I \otimes \Gamma_G)\Pi_G h &= (I \otimes W_j^*)(I \otimes \Gamma_G) \sum_{\mathbf{k} \in \mathbb{Z}_+^r} \mathbf{z}^{\mathbf{k}} \otimes D_{\hat{T}_{1n}, G} \hat{T}_{1n}(G)^{* \mathbf{k}} h \\
&= (I \otimes W_j^*) \sum_{\mathbf{k} \in \mathbb{Z}_+^r} \mathbf{z}^{\mathbf{k}} \otimes (D_{\hat{T}_{1n}, G} \hat{T}_{1n}(G)^{* \mathbf{k}} h, D_{\hat{T}_1, G} \hat{T}_{1n}(G)^{* \mathbf{k}} T_1^* h) \\
&= \sum_{\mathbf{k} \in \mathbb{Z}_+^r} \mathbf{z}^{\mathbf{k}} \otimes (D_{\hat{T}_{1n}, G} \hat{T}_{1n}(G)^{* \mathbf{k}} T_j^* h, D_{\hat{T}_1, G} \hat{T}_{1n}(G)^{* \mathbf{k}} T_1^* T_j^* h) \\
&= (I \otimes \Gamma_G)\Pi_G T_j^* h.
\end{aligned}$$

Therefore,

$$(3.8) \quad (I \otimes \Gamma_G)\Pi_G T_j^* = V_j^*(I \otimes \Gamma_G)\Pi_G.$$

for all $j \in \bar{G}$. We now proceed towards finding contractions corresponding to $j \in G$. Since Π_G is the canonical dilation map of $\hat{T}_{1n}(G)$,

$$\Pi_G T_1^* T_n^* = M_{z_1}^* \Pi_G \text{ and } \Pi_G T_{m_i}^* = M_{z_i}^* \Pi_G \quad (1 < i \leq r),$$

where $(M_{z_1}, \dots, M_{z_r})$ is the r -tuple of shifts on $H_{\mathcal{D}_{\hat{T}_{1n}, G}}^2(\mathbb{D}^r)$. Then setting

$$V_{m_i} := M_{z_i} \text{ on } H_{\mathcal{H}_G}^2(\mathbb{D}^r) \quad (1 < i \leq r)$$

and using the commuting property of $M_{z_i}^*$ and $(I \otimes \Gamma_G)$ we have,

$$(I \otimes \Gamma_G)\Pi_G T_1^* T_n^* = M_{z_1}^*(I \otimes \Gamma_G)\Pi_G \text{ and } (I \otimes \Gamma_G)\Pi_G T_{m_i}^* = V_{m_i}^*(I \otimes \Gamma_G)\Pi_G \quad (1 < i \leq r).$$

Next we factor M_{z_1} , using Lemma 3.2, to obtain contractions corresponding to $j = 1, n$ as follows. Recall that $\mathcal{K}_G = \mathcal{K}_{n, G} \oplus \mathcal{K}_{1, G} \oplus \mathcal{K}$. Let $P : \mathcal{K}_G \rightarrow \mathcal{K}_{1, G} \oplus \mathcal{K}$ be the projection map and $\iota_1 : \mathcal{K}_{n, G} \hookrightarrow \mathcal{K}_G$ and $\iota_2 : \mathcal{K}_{1, G} \oplus \mathcal{K} \hookrightarrow \mathcal{K}_G$ be the inclusion maps defined by

$$\iota_1(h) = (h, 0, 0), \text{ and } \iota_2(k, k') = (0, k, k'), \quad (h \in \mathcal{K}_{n, G}, k \in \mathcal{K}_{1, G}, k' \in \mathcal{K}).$$

Then it is easy to see that

$$\begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} : \mathcal{K}_G \oplus \mathcal{K}_{n, G} \rightarrow \mathcal{K}_G \oplus \mathcal{K}_{n, G},$$

is a unitary, and therefore

$$U_1 = \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P & \iota_1 \\ \iota_1^* & 0 \end{bmatrix} = \begin{bmatrix} U^* P & U^* \iota_1 \\ \iota_1^* & 0 \end{bmatrix} : \mathcal{K}_G \oplus \mathcal{K}_{n, G} \rightarrow \mathcal{K}_G \oplus \mathcal{K}_{n, G}$$

is also a unitary, where U is the unitary as in (3.7). We now claim that U_1 satisfies the hypothesis of Lemma 3.2 with $T = \hat{T}_{1n}(G)$ and $F_1 = D_{\hat{T}_{1n}, G}$. Indeed, for $h \in \mathcal{H}$,

$$\begin{aligned}
U_1(\Gamma_G D_{\hat{T}_{1n}, G} h, D_{\hat{T}_{1n}, G} T_1^* T_n^* h) &= U_1(D_{\hat{T}_{1n}, G} h, D_{\hat{T}_1, G} T_1^* h, 0_{\mathcal{K}}, D_{\hat{T}_{1n}, G} T_1^* T_n^* h) \\
&= (U^*(D_{\hat{T}_{1n}, G} T_1^* T_n^* h, D_{\hat{T}_1, G} T_1^* h, 0_{\mathcal{K}}), D_{\hat{T}_{1n}, G} h) \\
&= (D_{\hat{T}_{1n}, G} T_1^* h, D_{\hat{T}_1, G} T_1^{*2} h, 0_{\mathcal{K}}, D_{\hat{T}_{1n}, G} h) \\
&= (\Gamma_G D_{\hat{T}_{1n}, G} T_1^* h, D_{\hat{T}_{1n}, G} h).
\end{aligned}$$

Similarly, one can check that the unitary

$$U_2 = \begin{bmatrix} P^\perp & \iota_2 \\ \iota_2^* & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P^\perp U & \iota_2 \\ \iota_2^* U & 0 \end{bmatrix} : \mathcal{K}_G \oplus (\mathcal{K}_{1,G} \oplus \mathcal{K}) \rightarrow \mathcal{K}_G \oplus (\mathcal{K}_{1,G} \oplus \mathcal{K}),$$

satisfies

$$U_2(\Gamma_G D_{\hat{T}_{1n,G}} h, D_{\hat{T}_{1,G}} T_1^* T_n^* h, 0_\mathcal{K}) = (\Gamma_G D_{\hat{T}_{1n,G}} T_n^* h, D_{\hat{T}_{1,G}} h, 0_\mathcal{K}),$$

for all $h \in \mathcal{H}$. Therefore by Lemma 3.2, we have

$$(I \otimes \Gamma_G) \Pi_G T_1^* = M_\Phi^* (I \otimes \Gamma_G) \Pi_G \text{ and } (I \otimes \Gamma_G) \Pi_G T_n^* = M_\Psi^* (I \otimes \Gamma_G) \Pi_G,$$

where

$$\Phi(\mathbf{z}) = (P + z_1 P^\perp) U \text{ and } \Psi(\mathbf{z}) = U^* (P^\perp + z_1 P), \quad (\mathbf{z} \in \mathbb{D}^r)$$

are inner multipliers, depend only on z_1 variable, with

$$\Phi(\mathbf{z}) \Psi(\mathbf{z}) = \Psi(\mathbf{z}) \Phi(\mathbf{z}) = z_1 I_{\mathcal{K}_G},$$

for all $\mathbf{z} \in \mathbb{D}^r$. Since $\overline{\text{ran}}(I \otimes \Gamma_G) \Pi_G \subseteq H_{\mathcal{H}_G}^2(\mathbb{D}^r)$, we also have

$$(I \otimes \Gamma_G) \Pi_G T_1^* = V_1^* (I \otimes \Gamma_G) \Pi_G \text{ and } (I \otimes \Gamma_G) \Pi_G T_n^* = V_n^* (I \otimes \Gamma_G) \Pi_G,$$

where

$$V_1 = P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_\Phi|_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} \text{ and } V_n = P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_\Psi|_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)}.$$

We pause for a moment and make a remark that even though M_Φ and M_Ψ commute each other, V_1 and V_n does not necessarily commute. The reader must have observed that we have obtained the n -tuple of contractions (V_1, \dots, V_n) on $H_{\mathcal{H}_G}^2(\mathbb{D}^r)$ with the required intertwining property. The proof will be complete if we show that \hat{V}_1 and \hat{V}_n are doubly commuting. We only show that \hat{V}_n is doubly commuting as the proof for \hat{V}_1 is similar.

Since M_Φ and M_{z_i} on $H_{\mathcal{K}_G}^2(\mathbb{D}^r)$ doubly commute each other for all $i = 2, \dots, r$, it follows that V_1 and V_{m_i} doubly commute for all $i = 2, \dots, r$. It is obvious that V_{m_i} doubly commutes with V_j for all $i = 2, \dots, r$ and $j \in \bar{G}$. Thus it remains to show that V_1 commutes with V_j for all $j \in \bar{G}$. To this end, we first claim that

$$P_{\mathcal{H}_G}(PU)W_j^* = W_j^* P_{\mathcal{H}_G}(PU|_{\mathcal{H}_G}) \text{ and } P_{\mathcal{H}_G}(P^\perp U)W_j^* = W_j^* P_{\mathcal{H}_G}(P^\perp U|_{\mathcal{H}_G})$$

for all $j \in \bar{G}$. Indeed, since \mathcal{H}_G is a joint $\{W_j^* : j \in \bar{G}\}$ reducing subspace then $P_{\mathcal{H}_G}$ commutes with W_j^* for all $j \in \bar{G}$. Also by the construction

$$PW_j^*|_{\mathcal{K}'_G} = W_j^* P|_{\mathcal{K}'_G}.$$

Then, using the intertwining property (3.6), we have for all $h \in \mathcal{H}_G$ and $j \in \bar{G}$,

$$P_{\mathcal{H}_G}(PU)W_j^* h = P_{\mathcal{H}_G}(PU'')W_j^* h = P_{\mathcal{H}_G}PW_j^* U'' h = W_j^* P_{\mathcal{H}_G}PU h = W_j^* P_{\mathcal{H}_G}PU h.$$

This proves the first identity in the claim and the proof of the second identity is similar and we left it for the reader. These identities, in turn, implies that

$$\begin{aligned}
P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_{\Phi} |_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} (I \otimes W_j^*) |_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} &= P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_{(P+z_1 P^\perp)U} (I \otimes W_j^*) |_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} \\
&= P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_{(PUW_j^*+z_1 P^\perp U W_j^*)} |_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} \\
&= P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} (I \otimes W_j^*) P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_{(PU+z_1 P^\perp U)} |_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} \\
&= (I \otimes W_j^*) P_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)} M_{\Phi} |_{H_{\mathcal{H}_G}^2(\mathbb{D}^r)}.
\end{aligned}$$

Thus V_1 commutes with V_j^* and consequently, by Fuglede-Putnam, V_1 commutes with V_j for all $j \in \bar{G}$. This completes the proof. \square

Remark 3.4. *It should be noted that the canonical dilation map Π_G is not an isometry and that is why the n -tuple of contractions V is not a co-extension of T . A word of caution is in order regarding the n -tuple contractions $V = (V_1, \dots, V_n)$. The tuple V is not a commuting tuple, in general, and the only trouble is that V_1 and V_n do not commute each other. However, $(V_0 = M_{z_1}, V_1, \dots, V_{n-1})$ on $H_{\mathcal{H}_G}^2(\mathbb{D}^r)$ is an n -tuple of commuting contractions with*

$$(I \otimes \Gamma_G) \Pi_G T_1^* T_n^* = M_{z_1}^* (I \otimes \Gamma_G) \Pi_G, \text{ and } (I \otimes \Gamma_G) \Pi_G T_i^* = V_i^* (I \otimes \Gamma_G) \Pi_G \text{ (} 1 \leq i \leq n-1 \text{),}$$

and also

$$(3.9) \quad V_1^* V_0 = V_n \text{ and } V_n^* V_0 = V_1.$$

Similar statement also can be made for the tuple $(V_0 = M_{z_1}, V_2, \dots, V_n)$.

The situation for the case when $1 \notin G$ is much simpler and we consider it in the next lemma.

Lemma 3.5. *Let $T = (T_1, \dots, T_n) \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. Let $G = \{m_1, \dots, m_r\} \subset \{1, \dots, n-1\}$ with $1 \notin G$ and set $\bar{G} = \{1, \dots, n-1\} \setminus G$. Suppose T_j is a co-isometry for all $j \in \bar{G} \cup \{n\}$. Then there exist a Hilbert space $\mathcal{H}_G \supset \mathcal{D}_{\hat{T}_{1n}, G}$ and an n -tuple of commuting isometries $V = (V_1, \dots, V_n)$ on $H_{\mathcal{H}_G}^2(\mathbb{D}^r)$ such that \hat{V}_1 and \hat{V}_n are doubly commuting and*

$$\Pi_G T_i^* = V_i^* \Pi_G \text{ (} 1 \leq i \leq n \text{)}$$

where $\Pi_G : \mathcal{H} \rightarrow H_{\mathcal{D}_{\hat{T}_{1n}, G}}^2(\mathbb{D}^r) \subset H_{\mathcal{H}_G}^2(\mathbb{D}^r)$ is the canonical dilation map of $\hat{T}_{1n}(G)$ and

$$V_{m_i} = M_{z_i} \text{ (} 1 \leq i \leq r \text{) and } V_j = I \otimes U_j \text{ (} j \in \bar{G} \cup \{n\} \text{)}$$

for some commuting unitaries U_j 's on \mathcal{H}_G .

Proof. Since T_j ($j \in \bar{G}$) and T_n are co-isometries, a straight forward computation as done in the proof of Lemma 3.3 yields

$$T_j D_{T,G}^2 T_j^* = D_{T,G}^2 \text{ and } T_n D_{T,G}^2 T_n^* = D_{T,G}^2.$$

Then, by Douglas' lemma, there exist co-isometries $S_j : \mathcal{D}_{T,G} \rightarrow \mathcal{D}_{T,G}$ ($j \in \bar{G}$) and $S_n : \mathcal{D}_{T,G} \rightarrow \mathcal{D}_{T,G}$ such that

$$S_j^* D_{T,G} h = D_{T,G} T_j^* h \text{ and } S_n^* D_{T,G} h = D_{T,G} T_n^* h,$$

for all $h \in \mathcal{H}$. Clearly $\{S_j : j \in \bar{G} \cup \{n\}\}$ is a tuple of commuting co-isometries on $\mathcal{D}_{T,G}$. Let the commuting tuple of unitaries $\{U_j : j \in \bar{G} \cup \{n\}\}$ on $\mathcal{H}_G \supseteq \mathcal{D}_{T,G}$ be the minimal unitary co-extension of $\{S_j : j \in \bar{G} \cup \{n\}\}$. On the other hand, since $\Pi_G : \mathcal{H} \rightarrow H_{\mathcal{D}_{T,G}}^2(\mathbb{D}^r) \subset H_{\mathcal{H}_G}^2(\mathbb{D}^r)$ is the canonical dilation map of $\hat{T}_{1n}(G)$ then

$$\Pi_G T_{m_i}^* = M_{z_i}^* \Pi_G$$

for all $i = 1 \dots, r$. Also it is evident from the construction of U_j 's that

$$\Pi_G T_j^* = (I \otimes U_j)^* \Pi_G$$

for all $j \in \bar{G} \cup \{n\}$. Set

$$V_{m_i} := M_{z_i} \quad (1 \leq i \leq r) \quad \text{and} \quad V_j := I \otimes U_j \quad (j \in \bar{G} \cup \{n\}).$$

Then the n -tuple of commuting isometries $V = (V_1, \dots, V_n)$ has the required property. This completes the proof. \square

We need one more lemma which describes a canonical way to construct co-isometries out of commuting contractions.

Lemma 3.6. *Let $T = (T_1, \dots, T_n) \in \mathcal{T}^n(\mathcal{H})$. Let $G \subset \{1, \dots, n\}$ and $\bar{G} = \{1, \dots, n\} \setminus G$. Then there exist a positive operator $Q : \mathcal{H} \rightarrow \mathcal{H}$ and contractions $\tilde{T}_j : \overline{\text{ran}} Q \rightarrow \overline{\text{ran}} Q$ ($1 \leq j \leq n$) defined by*

$$\tilde{T}_j^* Q h = Q T_j^* h, \quad (h \in \mathcal{H})$$

such that \tilde{T}_j is a co-isometry for all $j \in \bar{G}$.

Proof. Since $T_{\bar{G}}$ is a contraction, strong operator limit (SOT) of $T_{\bar{G}}^n T_{\bar{G}}^{*n}$ exists as $n \rightarrow \infty$. Set

$$Q^2 := \text{SOT-} \lim_{n \rightarrow \infty} T_{\bar{G}}^n T_{\bar{G}}^{*n}.$$

Since T_j is a contraction, it is easy to see that

$$T_j Q^2 T_j^* \leq Q^2,$$

for all $j = 1 \dots, n$. Consequently by Douglas' lemma, for each $1 \leq j \leq n$, there exists a contraction $\tilde{T}_j : \overline{\text{ran}} Q \rightarrow \overline{\text{ran}} Q$ such that

$$\tilde{T}_j^* Q h = Q T_j^* h \quad (h \in \mathcal{H}).$$

Moreover, for all $j \in \bar{G}$,

$$Q^2 \geq T_j Q^2 T_j^* = \text{SOT-} \lim_{n \rightarrow \infty} T_{\bar{G}}^n T_j T_j^* T_{\bar{G}}^{*n} \geq \text{SOT-} \lim_{n \rightarrow \infty} T_{\bar{G}}^{n+1} T_{\bar{G}}^{*(n+1)} = Q^2,$$

that is

$$T_j Q^2 T_j^* = Q^2.$$

Hence \tilde{T}_j is a co-isometry for all $j \in \bar{G}$. This completes the proof. \square

Combining the above lemmas we now find co-extensions for n -tuples in $\mathfrak{B}_{1,n}^n(\mathcal{H})$.

Proposition 3.7. *Let $T = (T_1, \dots, T_n) \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. Then for each $G \subseteq \{1, \dots, n-1\}$ there exist a Hilbert space \mathcal{H}_G and contractions $V_{G,1}, \dots, V_{G,n}$ on $H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ such that*

$$\Pi T_j^* = \left(\bigoplus_G V_{G,j}^* \right) \Pi, \quad (1 \leq j \leq n)$$

where $\Pi : \mathcal{H} \rightarrow \bigoplus_G H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ is an isometry and by convention $H_{\mathcal{H}_0}^2(\mathbb{D}^{|\emptyset|}) := \mathcal{H}_0$.

Moreover, if we set $V_j := \bigoplus_G V_{G,j}$ for all $j = 1, \dots, n$ and $V = (V_1, \dots, V_n)$ then \hat{V}_1 and \hat{V}_n are doubly commuting and V_j is an isometry for all $j = 2, \dots, n-1$.

Proof. As usual, we shall work on the $(n-1)$ -tuple $\hat{T}_{1n} \in \mathfrak{B}^{n-1}(\mathcal{H})$. Let us fix $G = \{m_1, \dots, m_{|G|}\} \subseteq \{1, \dots, n-1\}$ and set $\bar{G} = \{1, \dots, n-1\} \setminus G$. Let $Q_G : \mathcal{H} \rightarrow \mathcal{H}$ be the positive operator defined by

$$Q_G^2 := \text{SOT-} \lim_{m \rightarrow \infty} (\hat{T}_{1n}(\bar{G}))^m (\hat{T}_{1n}(\bar{G}))^{*m}.$$

Then, by Lemma 3.6, we have a contraction $S_j : \overline{\text{ran}} Q_G \rightarrow \overline{\text{ran}} Q_G$ defined by

$$S_j^* Q_G h = Q_G T_j^* h, \quad (h \in \mathcal{H})$$

for all $j = 1, \dots, n$ so that S_j is a co-isometry for all $j \in \bar{G}$. In the case when $1 \in \bar{G}$, both S_1 and S_n are co-isometries because their product $S_1 S_n$ is a co-isometry. We claim that

$$S := (S_1, \dots, S_n) \in \mathfrak{B}_{1,n}^n(\overline{\text{ran}} Q_G).$$

Indeed, the claim follows from the fact that if $\sum_{F \subset G'} (-1)^{|F|} (\hat{T}_{1n})_F (\hat{T}_{1n})_F^* \geq 0$ for some $G' \subset \{1, \dots, n-1\}$ then

$$\sum_{F \subset G'} (-1)^{|F|} (\hat{T}_{1n})_F Q_G^2 (\hat{T}_{1n})_F^* = \text{SOT-} \lim_{m \rightarrow \infty} \hat{T}_{1n}(\bar{G})^m \left(\sum_{F \subset G'} (-1)^{|F|} (\hat{T}_{1n})_F (\hat{T}_{1n})_F^* \right) \hat{T}_{1n}(\bar{G})^{*m} \geq 0,$$

and the later positivity is equivalent to the positivity of $\sum_{F \subset G'} (-1)^{|F|} (\hat{S}_{1n})_F (\hat{S}_{1n})_F^*$. Next we apply Lemma 3.3 and Lemma 3.5 for the tuple $S := (S_1, \dots, S_n) \in \mathfrak{B}_{1,n}^n(\overline{\text{ran}} Q_G)$, to obtain the building blocks of the co-extension we are after. We consider the following two cases.

Case I: Suppose $1 = m_1 \in G$. Since S_j ($j \in \bar{G}$) are co-isometries, then by Lemma 3.3, there exist a Hilbert space \mathcal{H}_G , an isometry $\Gamma_G : \mathcal{D}_{\hat{S}_{1n}, G} \rightarrow \mathcal{H}_G$ and n -tuple of contractions $V = (V_{G,1}, \dots, V_{G,n})$ on $H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ such that \hat{V}_1 and \hat{V}_n are doubly commuting and

$$(I \otimes \Gamma_G) \tilde{\Pi}_G S_i^* = V_{G,i}^* (I \otimes \Gamma_G) \tilde{\Pi}_G, \quad (1 \leq i \leq n)$$

where $\tilde{\Pi}_G : \overline{\text{ran}} Q_G \rightarrow H_{\mathcal{D}_{\hat{S}_{1n}, G}}^2(\mathbb{D}^{|G|})$ is the canonical dilation map of $\hat{S}_{1n}(G)$. Now using the identity $S_j^* Q_G = Q_G T_j^*$ and setting

$$\Pi_G := \tilde{\Pi}_G Q_G,$$

we have that

$$(3.10) \quad (I \otimes \Gamma_G) \Pi_G T_i^* = V_{G,i}^* (I \otimes \Gamma_G) \Pi_G, \quad (1 \leq i \leq n)$$

where $(I \otimes \Gamma_G)\Pi_G : \mathcal{H} \rightarrow H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ is a contraction with

$$\begin{aligned} \|(I \otimes \Gamma_G)\Pi_G h\|^2 &= \lim_{k \rightarrow \infty} \sum_{F \subset G} (-1)^{|F|} \|(\hat{S}_{1n})_F^{*k} Q_G h\|^2 \\ &= \lim_{k \rightarrow \infty} \sum_{F \subset G} (-1)^{|F|} \|Q_G(\hat{T}_{1n})_F^{*k} h\|^2, \end{aligned}$$

for all $h \in \mathcal{H}$.

Case II: Suppose $1 \notin G$. In such a case, S_j is a co-isometry for all $j \in \bar{G} \cup \{n\}$ and therefore, by Lemma 3.5, there exist a Hilbert space \mathcal{H}_G and commuting n -tuple of isometries $V' = (V_{G,1}, \dots, V_{G,n})$ on $H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ such that \hat{V}'_1 and \hat{V}'_n are doubly commuting and

$$\tilde{\Pi}_G S_i^* = V_{G,i}^* \tilde{\Pi}_G \quad (1 \leq i \leq n)$$

where $\tilde{\Pi}_G : \overline{\text{ran}} Q_G \rightarrow H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ is the canonical dilation map of $\hat{S}_{1n}(G)$. Again using the identity $S_j^* Q_G = Q_G T_j^*$ and setting

$$\Pi_G := \tilde{\Pi}_G Q_G,$$

we get

$$(3.11) \quad \Pi_G T_i^* = V_{G,i}^* \Pi_G, \quad (1 \leq i \leq n)$$

and for all $h \in \mathcal{H}$,

$$\|\Pi_G h\|^2 = \lim_{k \rightarrow \infty} \sum_{F \subset G} (-1)^{|F|} \|Q_G(\hat{T}_{1n})_F^{*k} h\|^2.$$

Now we combine all the intertwining maps, obtained in the above two cases, together to obtain a co-extension. Define $\Pi : \mathcal{H} \rightarrow \bigoplus_G H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ by

$$\Pi(h)(G) = \begin{cases} (I \otimes \Gamma_G)\Pi_G h & 1 \in G \\ \Pi_G h & \text{otherwise} \end{cases}, \quad (h \in \mathcal{H}, G \subset \{1, \dots, n-1\})$$

where $\Pi(h)(G)$ denotes the G -th coordinate of $\Pi(h)$. Then by (3.10) and (3.11), it follows that

$$\Pi T_j^* = \left(\bigoplus_G V_{G,j}^* \right) \Pi$$

for all $j = 1, \dots, n$. Note that if Π is an isometry then the above identity gives a co-extension of T . To show Π is an isometry, for any $h \in \mathcal{H}$, we compute

$$\begin{aligned} \|\Pi h\|^2 &= \sum_{G \subset \{1, \dots, n-1\}} \|\Pi_G h\|^2 \quad (\text{as } \Gamma_G \text{'s are isometry}) \\ &= \sum_{G \subset \{1, \dots, n-1\}} \lim_{k \rightarrow \infty} \sum_{F \subset G} (-1)^{|F|} \|Q_G(\hat{T}_{1n})_F^{*k} h\|^2 \\ &= \sum_{G \subset \{1, \dots, n-1\}} \sum_{F \subset G} (-1)^{|F|} \lim_{k \rightarrow \infty} \|T_{\bar{G}}^{*k}(\hat{T}_{1n})_F^{*k} h\|^2 \\ &= \sum_{A \subset \{1, \dots, n-1\}} \lim_{k \rightarrow \infty} \|T_A^{*k} h\|^2 \sum_{F \subset A} (-1)^{|F|}. \end{aligned}$$

For each subset $A \neq \emptyset$ one can see that $\sum_{F \subset A} (-1)^{|F|} = 0$. So $\|\Pi h\|^2 = \|h\|^2$ for all $h \in \mathcal{H}$ and hence Π is an isometry. The moreover part is now clear from the construction of $V_{G,j}$'s. This completes the proof. \square

We make several important remarks about the above proposition and these observations will be used to prove the main theorem below.

Remarks 3.8. (i) *The main drawback of the above proposition is that the co-extension V of T is not a commuting tuple of contractions (see Remark 3.4). As noted earlier, the only problem is that V_1 and V_n do not commute each other. However, there are several things which are nice and help us to work further to find isometric dilation of T . For example, V_j is an isometry for all $j = 2, \dots, n-1$ and if we set*

$$V_0 = \bigoplus_G V_{G,0} \in \mathcal{B}\left(\bigoplus_G H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})\right), \text{ where } V_{G,0} = \begin{cases} M_{z_1} & 1 \in G \\ V_{G,1}V_{G,n} & \text{otherwise} \end{cases},$$

then V_0 is an isometry, $(V_0, V_1, \dots, V_{n-1})$ is an n -tuple of commuting contractions with

$$\Pi T_1^* T_n^* = V_0^* \Pi \text{ and } \Pi T_j^* = V_j^* \Pi \quad (1 \leq j \leq n-1).$$

and, in view of (3.9),

$$(3.12) \quad V_1^* V_0 = V_n \text{ and } V_n^* V_0 = V_1.$$

Another crucial fact is that (V_1, \dots, V_{n-1}) and $(V_0, V_2, \dots, V_{n-1})$ are doubly commuting. In other words, the doubly commuting tuple (V_2, \dots, V_{n-1}) doubly commute with both V_0 and V_1 . Needless to say that a similar statement also can be made about the commuting tuple of contractions (V_0, V_2, \dots, V_n) .

(ii) If $\mathcal{Q} := \text{ran } \Pi$, then \mathcal{Q} is a joint $(V_0^*, V_1^*, \dots, V_n^*)$ -invariant subspace of $\bigoplus_G H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$, $(P_{\mathcal{Q}}V_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}V_n|_{\mathcal{Q}})$ is an n -tuple of commuting contractions and

$$(T_1, \dots, T_n) \cong (P_{\mathcal{Q}}V_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}V_n|_{\mathcal{Q}}).$$

Moreover,

$$(P_{\mathcal{Q}}V_1|_{\mathcal{Q}})(P_{\mathcal{Q}}V_n|_{\mathcal{Q}}) = (P_{\mathcal{Q}}V_n|_{\mathcal{Q}})(P_{\mathcal{Q}}V_1|_{\mathcal{Q}}) = P_{\mathcal{Q}}V_0|_{\mathcal{Q}}.$$

We now find isometric dilations of operator tuples in $\mathfrak{B}_{1,n}^n(\mathcal{H})$, which is the main theorem of this article.

Theorem 3.9. *Let $T = (T_1, \dots, T_n) \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. Then T has an isometric dilation $W = (W_1, \dots, W_n)$ such that \hat{W}_1 and \hat{W}_n are $*$ -regular isometric dilations of \hat{T}_1 and \hat{T}_n , respectively.*

Proof. Let $\Pi, (V_{G,1}, \dots, V_{G,n})$ on $H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$ and $V = (V_1, \dots, V_n)$ be as in Proposition 3.7. Then $V = (V_1, \dots, V_n)$ is a co-extension of T , that is

$$\Pi T_j^* = V_j^* \Pi \quad (1 \leq j \leq n)$$

where $V_j = \bigoplus_G V_{G,j}$ for all $j = 1, \dots, n$. We set

$$V_0 := \bigoplus_G V_{G,0} \in \mathcal{B}\left(\bigoplus_G H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})\right), \text{ where } V_{G,0} = \begin{cases} M_{z_1} & 1 \in G \\ V_{G,1}V_{G,n} & \text{otherwise} \end{cases}.$$

As we have noted in Remark 3.8, $(V_0, V_1, \dots, V_{n-1})$ is a commuting tuple with V_i 's are isometry except V_1 and the doubly commuting tuple (V_2, \dots, V_{n-1}) doubly commute with both V_0 and V_1 . Since V_0 is an isometry, then the pair (V_0, V_1) has a regular unitary dilation (W_0, W_1) on \mathcal{K} . Then by Lemma 2.1, we extends V_j to an isometry W_j on \mathcal{K} , for all $j = 2, \dots, n-1$, such that the tuple $(W_0, W_1, \dots, W_{n-1})$ on \mathcal{K} is an isometric dilation of $(V_0, V_1, \dots, V_{n-1})$ on $\bigoplus_G H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$. Set

$$W_n := W_1^* W_0.$$

Then clearly (W_1, \dots, W_n) is a commuting tuple of isometries. We claim that (W_1, \dots, W_n) is an isometric dilation of $(P_{\mathcal{Q}}V_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}V_n|_{\mathcal{Q}})$, where $\mathcal{Q} = \text{ran } \Pi$. To prove the claim, let $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ and let us denote $\mathcal{X} := \bigoplus_G H_{\mathcal{H}_G}^2(\mathbb{D}^{|G|})$. We divide the proof of the claim in the following two cases.

Case I: If $k_n \geq k_1$, then

$$\begin{aligned} P_{\mathcal{X}} W_1^{k_1} \dots W_n^{k_n} |_{\mathcal{X}} &= P_{\mathcal{X}} W_0^{k_n} W_1^{*(k_n - k_1)} \dots W_{n-1}^{k_{n-1}} |_{\mathcal{X}} \\ &= (P_{\mathcal{X}} W_1^{*(k_n - k_1)} W_0^{k_n} |_{\mathcal{X}}) (W_2^{k_2} \dots W_{n-1}^{k_{n-1}} |_{\mathcal{X}}) \quad [W_j \mathcal{X} \subset \mathcal{X}, j = 2, \dots, n-1] \\ &= V_1^{*(k_n - k_1)} V_0^{k_n} V_2^{k_2} \dots V_{n-1}^{k_{n-1}} \quad [(W_0, W_1) \text{ is a regular dilation of } (V_0, V_1)] \\ &= (V_1^* V_0)^{(k_n - k_1)} V_0^{k_1} V_2^{k_2} \dots V_{n-1}^{k_{n-1}} \\ &= V_n^{(k_n - k_1)} V_0^{k_1} V_2^{k_2} \dots V_{n-1}^{k_{n-1}}, \quad [\text{by (3.12)}] \end{aligned}$$

and therefore using the fact that $\mathcal{Q} \subset \mathcal{X}$ is a joint $(V_0^*, V_1^*, \dots, V_n^*)$ -invariant subspace

$$\begin{aligned} P_{\mathcal{Q}} W_1^{k_1} \dots W_n^{k_n} |_{\mathcal{Q}} &= P_{\mathcal{Q}} V_n^{(k_n - k_1)} V_0^{k_1} V_2^{k_2} \dots V_{n-1}^{k_{n-1}} |_{\mathcal{Q}} \\ &= (P_{\mathcal{Q}} V_n |_{\mathcal{Q}})^{(k_n - k_1)} (P_{\mathcal{Q}} V_0 |_{\mathcal{Q}})^{k_1} (P_{\mathcal{Q}} V_2 |_{\mathcal{Q}})^{k_2} \dots (P_{\mathcal{Q}} V_{n-1} |_{\mathcal{Q}})^{k_{n-1}} \\ &= (P_{\mathcal{Q}} V_1 |_{\mathcal{Q}})^{k_1} \dots (P_{\mathcal{Q}} V_n |_{\mathcal{Q}})^{k_n}. \end{aligned}$$

Here for the last equality we have used that $P_{\mathcal{Q}} V_0 |_{\mathcal{Q}} = (P_{\mathcal{Q}} V_1 |_{\mathcal{Q}}) (P_{\mathcal{Q}} V_n |_{\mathcal{Q}})$.

Case II: If $k_1 > k_n$, then

$$\begin{aligned} P_{\mathcal{X}} W_1^{k_1} \dots W_n^{k_n} |_{\mathcal{X}} &= P_{\mathcal{X}} W_0^{k_n} W_1^{k_1 - k_n} \dots W_{n-1}^{k_{n-1}} |_{\mathcal{X}} \\ &= V_0^{k_n} V_1^{k_1 - k_n} \dots V_{n-1}^{k_{n-1}}, \end{aligned}$$

and therefore,

$$\begin{aligned} P_{\mathcal{Q}} W_1^{k_1} \dots W_n^{k_n} |_{\mathcal{Q}} &= P_{\mathcal{Q}} V_0^{k_n} V_1^{k_1 - k_n} \dots V_{n-1}^{k_{n-1}} |_{\mathcal{Q}} \\ &= (P_{\mathcal{Q}} V_1 |_{\mathcal{Q}})^{k_1} \dots (P_{\mathcal{Q}} V_n |_{\mathcal{Q}})^{k_n}. \end{aligned}$$

This proves the claim. On the other hand, we have already observed in Remark 3.8 that

$$(T_1, \dots, T_n) \cong (P_{\mathcal{Q}} V_1 |_{\mathcal{Q}}, \dots, P_{\mathcal{Q}} V_n |_{\mathcal{Q}}).$$

Hence (W_1, \dots, W_n) is an isometric dilation of (T_1, \dots, T_n) . Since the tuple (W_2, \dots, W_{n-1}) is a co-extension of (V_2, \dots, V_{n-1}) and (V_2, \dots, V_{n-1}) is a co-extension of (T_2, \dots, T_{n-1}) , (W_2, \dots, W_{n-1}) is a co-extension of (T_2, \dots, T_{n-1}) . Finally since \hat{W}_1 and \hat{W}_n are tuples of

doubly commuting isometries, it follows that \hat{W}_1 and \hat{W}_n are $*$ -regular isometric dilation of \hat{T}_1 and \hat{T}_n , respectively. This completes the proof. \square

Few remarks are in order.

Remarks 3.10. (i) *The converse of the above theorem is true. That is, if $T \in \mathcal{T}^n(\mathcal{H})$ has an isometric dilation $W \in \mathcal{T}^n(\mathcal{K})$ so that \hat{W}_1 and \hat{W}_n are $*$ -regular isometric dilation of \hat{T}_1 and \hat{T}_2 respectively, then $T \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. This immediately follows from the fact that an operator tuple satisfies Brehmer positivity if and only if it has a $*$ -regular isometric dilation.*

(ii) *If $T \in \mathfrak{B}_{p,q}^n(\mathcal{H})$ then interchanging T_p with T_1 and T_q with T_n we can assume $T \in \mathfrak{B}_{1,n}^n(\mathcal{H})$. So, Theorem 3.9 provides dilations for tuples in $\mathfrak{B}_{p,q}^n(\mathcal{H})$.*

(iii) *If $T \in \mathfrak{B}_{1,n}^n(\mathcal{H})$ such that \hat{T}_n is pure then T is also a member of the class $\mathcal{T}_{1,n}^n(\mathcal{H})$ (see (1.1)) considered in [4]. Note that in this case \hat{T}_{1n} is also a pure tuple. Then the positive operator Q_G , defined in the proof of Proposition 3.7, is 0 for all $G \subsetneq \{1, \dots, n-1\}$ and $Q_G = I_{\mathcal{H}}$ for $G = \{1, \dots, n-1\}$. This implies $\Pi_G = 0$ for all $G \subsetneq \{1, \dots, n-1\}$ and Π_G is an isometry for $G = \{1, \dots, n-1\}$. Then it follows from Lemma 3.3 and Proposition 3.7 that the commuting tuple of isometries $(M_{\Phi}, M_{z_2}, \dots, M_{z_{n-1}}, M_{\Psi})$ on $H_{\mathcal{K}_G}^2(\mathbb{D}^{n-1})$ is a co-extension of T . Thus we recover the isometric dilations obtained in [4] for such tuples.*

We end the section with the von Neumann inequality for $\mathfrak{B}_{p,q}^n(\mathcal{H})$. Recall that, if an n -tuple of commuting contraction has an isometric dilation then it satisfies von Neumann inequality. So we have the following theorem as an immediate consequence of Theorem 3.9.

Theorem 3.11. *Let $T \in \mathfrak{B}_{r,q}^n(\mathcal{H})$ with $1 \leq r < q \leq n$. Then,*

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{z \in \mathbb{D}^n} |p(z)|$$

for all $p \in \mathbb{C}[z_1, \dots, z_n]$.

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